Error estimates for joint Tikhonov- and Lavrentiev-regularization of constrained control problems

Dirk A. Lorenz*   Arnd Rösch†

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Abstract

We consider joint Tikhonov- and Lavrentiev-regularization of control problems with pointwise control- and state-constraints. We derive error estimates for the error which is introduced by the Tikhonov regularization. With the help of this results we show, that if the solution of the unconstrained problem has no active constraints, the same holds for the Tikhonov-regularized solution if the regularization parameter is small enough and a certain source condition is fulfilled.

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1 Introduction

In this paper we consider problems that can be interpreted either as optimal control problems or identification problems (inverse problems). Let $D' \subset D$ be bounded domains in $\mathbf{R}^N$ $(N = 2,3)$, $U = L^2(D)$ and Y be a Hilbert space. For a compact and linear mapping $S : U \to Y$, an element $y_d \in Y$ and bounded measurable functions $b : D \to \mathbf{R}$ and $\psi : D' \to \mathbf{R}$ we consider the constrained minimization problem

$$\min \|Su - y_d\|^2 \quad \text{s.t.} \quad 0 \leq u \leq b \text{ a.e. on } D, \quad Su \leq \psi. \text{ a.e. on } D' \qquad (P)$$

The space Y is called data space in inverse problems and state space for control problems. Moreover U is considered as solutions space or control space. Our specific feature is the presence of two types of different inequality constraints. The first one

$$0 \leq u \leq b \text{ a.e. on } D$$

*Corresponding author, Institute for Analysis and Algebra, TU Braunschweig, D-38902 Braunschweig, Germany, d.lorenz@tu-braunschweig.de.
†Universität Duisburg-Essen, Fachbereich Mathematik, Forsthausweg 2, D-47057 Duisburg, Germany, arnd.roesch@uni-due.de.

describes a set of physically meaningful solutions in inverse problems or a set of admissible controls. The second one

$$Su \leq \psi. \text{ a.e. on } D'$$

ensures that the data (the state) is pointwise in a reasonable range. In the inverse-problem context, the constraints on u and Su model prior knowledge on the solution and the measured data, respectively and hence, they shall lead to more reliable reconstructions.

Before we start to analyze this problem, we will give a specific example. Let $S : u \mapsto y$ be the solution operator of the boundary value problem

$$-\Delta y = u \text{ on } D, \quad y = 0 \text{ on } \partial D$$

with a Lipschitz domain D. We specify the spaces $U = Y = L^2(D)$. The set $D' \subset D$ can be for instance an inner subdomain. However, it would also be possible to discuss sets D' containing only finitely many points. Of course our setting fits also to more challenging problems.

Next, we will discuss the regularization of such a problem. Let us first mention that the introduced problem (P) can be well-posed or ill-posed. One can easily construct situations, where the set of functions u satisfying the inequality constraints is empty or consists of exactly one point. However, such situations are not in our focus. We will assume later that the set of feasible functions $u \in U$ has an inner point with respect to the L^∞-topology.

Since the set of feasible functions u is weakly compact in U, the existence of at least one function u satisfying the inequality constraints ensures the existence of a solution of (P). Consequently, existence of solutions of (P) will not be an important issue in this paper.

Let us now think about uniqueness of solutions. Using standard arguments one can show the uniqueness of the state (resp. data) $y := Su$. However, the uniqueness of u is only guaranteed if S is injective. Of course, a standard Tikhonov regularization guarantees uniqueness of solutions of the regularized problems. Therefore, the discussion of uniqueness aspects seems to be complete. However, there is another uniqueness aspect occurring even in the case of finite dimensional spaces. Solution of minimization problems are analyzed and computed by means of optimality conditions. The most convenient form of optimality conditions includes Lagrange multipliers, i.e., dual or adjoint variables. It may happen that there exist a subset $D'' \subset D'$ where more than one inequality holds as equality. Typically this effect is connected with nonuniqueness of the dual variables on the corresponding set D''. For the construction of specific examples we refer to [6]. Let us mention that Lagrange multipliers associated with the inequality $Su \leq \psi$ are in general only Borel measures, see Casas [1]. This low regularity of the dual variables was the motivation in [10] to introduce a second Lavrentiev type regularization.

Motivated by this argumentation we will study in this paper a family of Tikhonov and Lavrentiev regularized problems:

$$\min \|Su - y_d\|^2 + \alpha\|u\|^2 \;\text{ s.t. } 0 \leq u \leq b, \text{ a.e. on } D, \lambda u + Su \leq \psi \text{ a.e. on } D' \tag{P_α^λ}$$

with $\alpha, \lambda \geq 0$. We denote with $\bar{u}$ a solution of (P) and with u_α^λ a solution of (P_α^λ). Note that the solution of (P_α^λ) is always unique if it exists.

Next, one can study a lot of different regularization errors. People coming from inverse problems usually study the solution behavior for $\alpha \downarrow 0$. Since the inverse theory heavily uses representation formulas, pointwise inequalities are undesired.

People coming from optimal control state that the Tikhonov term represents the costs for the control and assume that the parameter α is a given, fixed model parameter. Hence, it would be enough to study the behavior for $\lambda \downarrow 0$ for a fixed Tikhonov parameter $\alpha > 0$. However, even in the optimal-control context one may take the position that the problem (P) is the one which shall be solved and that the Tikhonov regularization is employed only to stabilize the problem.

Of course, error estimates for the whole regularization process are of high interest. However, this problem seems to be challenging. In this paper, we will only contribute a little bit in answering this question.

Let us summarize our plans. The problem (P) is the one which we want to solve. Therefore, we are interested in estimates for the error $\bar{u} - u_\alpha^\lambda$. We split this error into the *Tikhonov error* and the *Lavrentiev error*:

$$\|\bar{u} - u_\alpha^\lambda\| \leq \underbrace{\|\bar{u} - u_\alpha^0\|}_{\text{Tikhonov error}} + \underbrace{\|u_\alpha^0 - u_\alpha^\lambda\|}_{\text{Lavrentiev error}} .$$

Estimates for the Lavrentiev error can be found in [10, 2, 3, 9]. Here, we focus on estimates of the Tikhonov error. The analysis of this error is well developed in the framework on inverse problems for problems without inequality constraints. However, for the constrained case there are only few results [5, Section 5.4].

The paper is structured as follows: In Section 2 we will state some preliminary results. Error estimates for the constrained Tikhonov regularization are located in Section 3. The activity of inequality constraints is analyzed in Section 4. Error estimates for the Lavrentiev regularization can be found in Section 5. The verification of the assumptions for a distributed elliptic control problem and a Fredholm integral operator is presented in Section 6.

2 Preliminary results

First, we introduce notations for the admissible sets for (P) and (P_α^λ)

$$\begin{aligned} U_{\text{ad}} &= \{u \in U \mid 0 \leq u \leq b,\ Su \leq \psi\}, \\ {U_{\text{ad}}}^\lambda &= \{u \in U \mid 0 \leq u \leq b,\ \lambda u + Su \leq \psi\}. \end{aligned}$$

Note that, due to $0 \leq u$ we have ${U_{\text{ad}}}^\lambda \subset {U_{\text{ad}}}^\mu$ for $\lambda \geq \mu$. With this notation we reformulate

$$\min \|Su - y_d\|^2 \quad \text{s.t.} \quad u \in U_{\text{ad}} \tag{P}$$

$$\min \|Su - y_d\|^2 + \alpha\|u\|^2 \quad \text{s.t.} \quad u \in {U_{\text{ad}}}^\lambda. \tag{P_α^λ}$$

Now we state some preliminary results:

Lemma 2.1. *Let the operator S be linear and continuous. Then the sets U_{ad} and ${U_{\text{ad}}}^\lambda$ are closed, convex and bounded.*

Definition 2.2. The *constrained pseudo inverse* of an operator S with respect to a convex and closed set C is defined via

$$\|S_C^+(y_d)\| = \min\{\|u\| \mid u \in \operatorname*{argmin}_C \|Su - y_d\|^2\}.$$

In other words, $S_C^+(y_d)$ is the minimizing element of the residuum which has minimal norm.

The operator S_C^+ is a nonlinear operator with the following properties.

Lemma 2.3. *If C is non-empty, convex, closed and bounded, then*

$$D(S_C^+) = Y.$$

Proof. Note that C is weakly sequentially closed and hence a minimizing sequence of $\|Su - y_d\|^2$ in U has a weak accumulation point. By lower semicontinuity we see that this accumulation point is indeed a minimizer, and hence, the constrained pseudo inverse exists. □

The following well known proposition shows continuity in the Tikhonov parameter α:

Proposition 2.4. *u_α^λ depends continuously on α for $\alpha > 0$.*

Proof. We drop the superscript λ because it is fixed here. Consider u_α and u_β for $\alpha, \beta > 0$. Since λ is fixed, u_β is admissible for (P_α^λ) and vice versa. Hence, we can insert these elements in the corresponding variational inequalities and obtain

$$\langle S^*Su_\alpha + \alpha u_\alpha - S^*y_d, u_\beta - u_\alpha\rangle \geq 0$$
$$\langle S^*Su_\beta + \beta u_\beta - S^*y_d, u_\alpha - u_\beta\rangle \geq 0.$$

We get

$$\langle S^*Su_\alpha + \alpha u_\alpha - S^*Su_\beta - \beta u_\beta, u_\beta - u_\alpha\rangle \geq 0$$

and obtain

$$\|Su_\alpha - Su_\beta\|^2 \leq \langle \alpha u_\alpha - \beta u_\beta, u_\beta - u_\alpha\rangle \leq |\alpha - \beta|\|u_\alpha\|\|u_\beta - u_\alpha\| - \beta\|u_\beta - u_\alpha\|^2.$$

Since the left hand side is positive we get

$$\|u_\beta - u_\alpha\| \leq \frac{|\alpha - \beta|}{\beta}\|u_\alpha\|$$

and the right hand side converges to 0 for $\beta \to \alpha$. □

Finally, we state a characterization of the solution of (P_α^λ) by means of a projection formula:

Lemma 2.5 (Projection formula)**.** *The solution u_α^λ of (P_α^λ) is characterized by*

$$u_\alpha^\lambda = P_{U_{\mathrm{ad}}^\lambda}\Big(-\frac{S^*(Su_\alpha^\lambda - y_d)}{\alpha}\Big). \tag{1}$$

Proof. We rewrite (P_α^λ) with the help of the indicator function $I_{U_{\mathrm{ad}}{}^\lambda}$ as

$$\min_u \|Su - y_d\|^2 + \alpha\|u\|^2 + I_{U_{\mathrm{ad}}{}^\lambda}(u).$$

With the help of subgradient calculus we get from optimality of u_α^λ

$$-\frac{S^*(Su_\alpha^\lambda - y_d)}{\alpha} \in \partial(\|\cdot\|^2 + I_{U_{\mathrm{ad}}{}^\lambda})(u_\alpha^\lambda).$$

Since $\partial(\|\cdot\|^2 + I_{U_{\mathrm{ad}}{}^\lambda})^{-1} = P_{U_{\mathrm{ad}}{}^\lambda}$ (see, e.g. [11]) this gives

$$P_{U_{\mathrm{ad}}{}^\lambda}\Big(-\frac{S^*(Su_\alpha^\lambda - y_d)}{\alpha}\Big) = u_\alpha^\lambda.$$

□

3 Error estimates for constrained Tikhonov regularization

Now we establish error estimates for the term $\|\bar{u} - u_\alpha^0\|$. Resembling results can be found in [5]. Due to the structural differences between inverse problems and optimal control we state the result in our terminology with an explicit constant and present a full proof.

Theorem 3.1. *Let $\bar{u}$ be a solution of (P) and denote with $P_{U_{\mathrm{ad}}}$ the projection onto U_{ad}. Moreover, let the following source condition be fulfilled:*

$$\exists w \in Y: \ \bar{u} = P_{U_{\mathrm{ad}}}(S^*w).$$

Then it holds:

$$\|u_\alpha^0 - \bar{u}\| \leq \sqrt{\alpha}\|w\| + \frac{1}{\sqrt{\alpha}}\|S\bar{u} - y_d\| \tag{2}$$

$$\|Su_\alpha^0 - y_d\| \leq 2\alpha\|w\| + \|S\bar{u} - y_d\| \tag{3}$$

Proof. By definition we have $u_\alpha^0 \in U_{\mathrm{ad}}$ and $\bar{u} \in U_{\mathrm{ad}}$. Hence, by optimality of u_α^0 we have

$$\|Su_\alpha^0 - y_d\|^2 + \alpha\|u_\alpha^0\|^2 \leq \|S\bar{u} - y_d\|^2 + \alpha\|\bar{u}\|^2.$$

Rearranging yields

$$\|Su_\alpha^0 - y_d\|^2 + \alpha(\|u_\alpha^0\|^2 - \|\bar{u}\|^2) \leq \|S\bar{u} - y_d\|^2$$

which we extend to

$$\|Su_\alpha^0 - y_d\|^2 + \alpha(\|u_\alpha^0\|^2 - \|\bar{u}\|^2 - 2\langle S^*w, u_\alpha^0 - \bar{u}\rangle + 2\langle S^*w, u_\alpha^0 - \bar{u}\rangle) \leq \|S\bar{u} - y_d\|^2. \tag{4}$$

Since $\bar{u} = P_{U_{\mathrm{ad}}}S^*w$ we have for all $u \in U_{\mathrm{ad}}$

$$\langle S^*w - \bar{u}, u - \bar{u}\rangle \leq 0.$$

Using this with $u = u_\alpha^0$ we get

$$\|u_\alpha^0\|^2 - \|\bar{u}\|^2 - 2\langle S^*w, u_\alpha^0 - \bar{u}\rangle \geq \|u_\alpha^0\|^2 - \|\bar{u}\|^2 - 2\langle \bar{u}, u_\alpha^0 - \bar{u}\rangle = \|u_\alpha^0 - \bar{u}\|^2.$$

We further estimate from (4)

$$\begin{aligned}
&\|Su_\alpha^0 - y_d\|^2 + \alpha(\|u_\alpha^0 - \bar u\|^2 + 2\langle S^*w, u_\alpha^0 - \bar u\rangle) \le \|S\bar u - y_d\|^2\\
\Leftrightarrow\quad &\|Su_\alpha^0 - y_d\|^2 + 2\langle \alpha w, Su_\alpha^0 - y_d\rangle + \alpha\|u_\alpha^0 - \bar u\|^2 \le \|S\bar u - y_d\|^2 + 2\langle \alpha w, S\bar u - y_d\rangle.
\end{aligned}$$

Completing the squares by adding $\|\alpha w\|^2$ gives

$$\|Su_\alpha^0 - y_d + \alpha w\|^2 + \alpha\|u_\alpha^0 - \bar u\|^2 \le \|S\bar u - y_d + \alpha w\|^2.$$

On the one hand this leads to

$$\|u_\alpha^0 - \bar u\|^2 \le \tfrac{1}{\alpha}\|S\bar u - y_d + \alpha w\|^2$$

which gives by taking square roots and using the triangle inequality

$$\|u_\alpha^0 - \bar u\| \le \tfrac{1}{\sqrt{\alpha}}\|S\bar u - y_d\| + \sqrt{\alpha}\|w\|.$$

On the other hand we conclude

$$\|Su_\alpha^0 - y_d + \alpha w\|^2 \le \|S\bar u - y_d + \alpha w\|^2$$

which implies

$$\|Su_\alpha^0 - y_d\| \le \|S\bar u - y_d\| + 2\alpha\|w\|.$$

□

The estimate (2) motivates the following parameter choice: If y_d is not in the range, we see that the right hand side in (2) is smallest for

$$\alpha^* = \frac{\|S\bar u - y_d\|}{\|w\|}$$

and hence, is a reasonable choice for the choice of the Tikhonov parameter α if the quantities were known. Nonetheless, the error estimate (2) is useful for the determination of the total error $\|u_\alpha^\lambda - \bar u\|$.

4 Activity of the constraints

In this section we investigate the following situation: Assume that the optimal solution $\bar u$ of (P) has no active inequality constraints. That means that in fact we would get the same solution without imposing any inequality constraints. However, the formulation with additional inequality constraints is reasonable since a solution of an unconstrained inverse problem can violate these constraints for noisy data.

Now the question arises: Can we expect a solution without active constraints also for the purely Tikhonov-regularized problem for small regularization parameters? In fact this can be shown with the help of the estimates of Theorem 3.1:

Theorem 4.1. *Let $S : U = L^2(D) \to Y$ with dense range and assume that*

$$u_n \to u \text{ in } U \implies Su_n \to Su \text{ in } L^\infty(D') \tag{5}$$

$$y_n \rightharpoonup y \text{ in } Y \implies S^*y_n \to Sy \text{ in } L^\infty(D) \tag{6}$$

Let $\bar u$ be a solution of (P) such that for some $\tau > 0$ it holds that $\tau < \bar u < b - \tau$ and $S\bar u < \psi - \tau$ hold. Moreover, let there be $w \in Y$ such that $\bar u = S^*w$.

Then there exists $\alpha_0 > 0$ such that for every $\alpha < \alpha_0$ the solution u_α^0 of (P_α^0) also fulfills $\tau < u_\alpha^0 < b - \tau$ and $Su_\alpha^0 < \psi - \tau$.

Proof. Since $\bar{u}$ does not have active constraints and S has dense range, it holds $S\bar{u} = y_d$. To see this, note that $\bar{u}$ fulfills

$$\langle S^*(S\bar{u} - y_d), u - \bar{u}\rangle \geq 0 \quad \text{for all} \quad u \in U_{\text{ad}}$$

and let us assume that there exists $v \in U$ but $v \notin U_{\text{ad}}$, such that

$$\langle S^*(S\bar{u} - y_d), v - \bar{u}\rangle < 0. \tag{7}$$

We may approximate v by bounded functions and hence, there exists a bounded $\bar{v}$ such that

$$\langle S^*(S\bar{u} - y_d), \bar{v} - \bar{u}\rangle < 0.$$

Now observe that for $\theta > 0$ small enough, the function $v^\theta = \theta\bar{v} + (1-\theta)\bar{u}$ is in U_{ad} since $\bar{u}$ does not have active constraints. We see

$$\langle S^*(S\bar{u} - y_d), v^\theta - \bar{u}\rangle = \theta\langle S^*(S\bar{u} - y_d), \bar{v} - \bar{u}\rangle < 0$$

which contradicts the optimality of $\bar{u}$ for (P). Hence, (7) has to be fulfilled for all $v \in U$ and this shows that $\bar{u}$ is also a solution of the unconstrained problem. Since S has dense range, this optimal value for this problem is 0 which shows $S\bar{u} = y_d$.

Moreover, we have by assumption $\bar{u} = S^*w = P_{U_{\text{ad}}}S^*w$. We conclude from Theorem 3.1 that

$$\|u^0_\alpha - \bar{u}\| \leq \sqrt{\alpha}\|w\|.$$

This implies $u^0_\alpha \to \bar{u}$ in L^2 and by assumption we know that $Su^0_\alpha \to S\bar{u}$ in $L^\infty(D')$. Because of $S\bar{u} < \psi - \tau$, we find $Su^0_\alpha < \psi - \tau/2$ for sufficiently small α. Hence, we can use the projection formula (1) with $P_{[0,b]}$ instead of $P_{U_{\text{ad}}}$ for small α and get

$$u^0_\alpha = P_{[0,b]}(-S^*\tfrac{1}{\alpha}(Su^0_\alpha - y_d)).$$

Again from Theorem 3.1 we find

$$\|Su^0_\alpha - y_d\| \leq 2\alpha\|w\|.$$

Consequently, $\|(Su^0_\alpha - y_d)/\alpha\|$ is uniformly bounded in Y. Let us now take an arbitrary sequence $\{\alpha_n\}$ with $\alpha_n \to 0$ for n to ∞. Since $(Su^0_{\alpha_n} - y_d)/\alpha_n$ is uniformly bounded in Y, we can find a weakly convergent subsequence in Y denoted by the index n'. The assumption on S^* yields strong convergence of $u^0_{\alpha_{n'}}$ in $L^\infty(D)$. We already know that $u^0_{\alpha_{n'}}$ converges in $U = L^2(D)$ with limit $\bar{u}$. By a standard argumentation we obtain that $u^0_\alpha \to \bar{u}$ in $L^\infty(D)$ for $\alpha \to 0$. Consequently, the control constraints are also inactive for sufficiently small α. $\square$

Let us note that the first argumentation in the proof works also for the weaker assumption that $u_n \to u$ in $L^p(D)$ implies $Su_n \to Su$ in $L^\infty(D')$ for a sufficiently large p. However, the assumptions for the adjoint operator cannot be weakened. Thus, the practical benefit of that generalization is only small.

The above result is also of interest in the case of ill-posed problems. Here we may ask the question if additional constraints on the solution in the minimization of the Tikhonov functional may destroy the optimal convergence rate. As we will see below, this is not the case in our setting. Similar to Theorem 3.1 we can state the following result:

Theorem 4.2. *Let $\bar{u}$ be a solution of (P) with $S\bar{u} = y_d$ and let y^δ be such that $\|y^\delta - y_d\| \le \delta$. Moreover, let the following source condition be fulfilled:*

$$\exists w \in Y : \ \bar{u} = P_{U_{\mathrm{ad}}}(S^* w).$$

Then it holds for

$$u_\alpha^\delta = \operatorname*{argmin}_{u \in U_{\mathrm{ad}}} \|Su - y^\delta\|^2 + \alpha \|u\|^2 \tag{8}$$

that

$$\begin{aligned} \|u_\alpha^\delta - \bar{u}\| &\le \sqrt{\alpha}\|w\| + \frac{\delta}{\sqrt{\alpha}} \\ \|Su_\alpha^\delta - y_d\| &\le 2\alpha\|w\| + \delta \end{aligned}$$

It also holds that all constraints get inactive for small α if the true solution of noisy data does not have active constraints:

Theorem 4.3. *Let $S : U \to Y$ fulfill the conditions (5) and (6). Let $\bar{u}$ be a solution of (P) such that for some $\tau > 0$ it holds that $\tau < \bar{u} < b - \tau$ and $S\bar{u} < \psi - \tau$ holds. Moreover, let there be $w \in Y$ such that $\bar{u} = S^* w$ and let u_α^δ be defined by (8). Finally let $\alpha(\delta)$ be a parameter choice rule such that*

$$\alpha(\delta) \to 0, \quad \frac{\delta}{\alpha(\delta)} \to 0 \ \text{ for } \ \delta \to 0.$$

Then there exists $\delta_0 > 0$ such that for $\delta < \delta_0$ it holds that $\tau < u_{\alpha(\delta)}^\delta < b - \tau$ and $Su_{\alpha(\delta)}^\delta < \psi - \tau$ holds.

Proof. Due to the parameter choice we get $u_{\alpha(\delta)}^\delta \to \bar{u}$ strongly in L^2 for $\delta \to 0$. Similar to the proof of Theorem 4.1 we conclude that the state constraints are not active for sufficiently small δ and hence, the projection formula

$$u_{\alpha(\delta)}^\delta = P_{[0,b]}(-S^*(\tfrac{1}{\alpha(\delta)}(Su_{\alpha(\delta)}^\delta - y^\delta))$$

holds. Now the claim follows similarly to Theorem 4.1. □

Finally we state the following converse result which shows that the source condition $\bar{u} = P_{U_{\mathrm{ad}}} S^* w$ follows from weak convergence of the regularized solutions together with a mild decay of the discrepancy.

Theorem 4.4. *Let $u_\alpha^0 \rightharpoonup \bar{u}$ and $\|Su_\alpha^0 - y_d\| = O(\alpha)$ for $\alpha \to 0$. Moreover, we assume that S fulfills (5) and (6). Then there is a function w such that $P_{U_{\mathrm{ad}}} S^* w = \bar{u}$.*

Proof. Since $\|Su_\alpha^0 - y_d\| = O(\alpha)$ we have that

$$\limsup_{\alpha \to 0} \frac{\|Su_\alpha^0 - y_d\|}{\alpha} < \infty.$$

Hence there is a sequence $\alpha_n \to 0$ and an element w such that

$$\frac{Su_{\alpha_n} - y_d}{\alpha_n} \rightharpoonup -w.$$

By the projection formula (1) we have

$$P_{U_{\mathrm{ad}}}\Big(-\frac{S^*(Su^0_{\alpha_n}-y_d)}{\alpha_n}\Big)=u^0_{\alpha_n}.$$

For the right hand side converges weakly by assumption. With a discussion similar to that one in Theorem 4.1 we obtain the strong convergence of the left hand side of the last equation. Since the both sides converge weakly we have by uniqueness of the weak limit

$$P_{U_{\mathrm{ad}}}S^*w=\bar{u}.$$

□

5 Lavrentiev regularization

We discuss now the additional Lavrentiev regularization. The motivation of this second regularization is to improve the properties of the adjoint problem. In this section we will investigate two different situations. In the first part we will sketch the estimation of the Lavrentiev error in the general case. In the second one we will again discuss the situation of Theorem 4.1. Then we will be able to show better convergence results.

Let us start with the general case. The discussion for fixed $\alpha>0$ and a specific problem can be found in [3]. Next, we reformulate the assumptions of that paper for our more general setting.

$$S:U=L^2(D)\to L^\infty(D') \text{ continuously.} \tag{9}$$

$$\text{There exists } \hat{u}\in U,\tau>0 \text{ such that } 0\le\hat{u}\le b,\ S\hat{u}\le\psi-\tau. \tag{10}$$

The assumption (10) means that there exists a Slater point with respect to the state constraints. In the second part of this section we will benefit from the stronger assumption that $\bar{u}$ itself has this Slater property. In Section 2 we already mentioned that the set ${U_{\mathrm{ad}}}^\lambda$ of admissible u becomes smaller when λ becomes larger. However, one can show the existence of at least one admissible control for $\lambda\le\frac{\tau}{\|\hat{u}\|_{L^\infty(D')}}$.

The error estimates are obtained by the following technique:

1. Write down the necessary optimality conditions for u^0_α and u^λ_α as variational inequalities.

2. Take u^λ_α as test function in the optimality condition of u^0_α.

3. Construct a convex linear combination $u_\sigma=\sigma\hat{u}+(1-\sigma)u^0_\alpha$ which belongs to ${U_{\mathrm{ad}}}^\lambda$. Take this function as test function in the optimality condition of u^λ_α.

4. Add both inequalities and estimate all terms.

The resulting error estimate can be found in [3, Theorem 5.4]:

Lemma 5.1. *Let the assumptions (9) and (10) be fulfilled. Then there exists a constant $c > 0$ such that for $\lambda \le \frac{\tau}{\|\hat{u}\|_{L^\infty(D')}}$ the error of the Lavrentiev regularization can be estimated by*

$$\|u_\alpha^0 - u_\alpha^\lambda\| \le c\frac{\lambda}{\alpha}.$$

This general result has an essential drawback: If α becomes small, then λ has to be very small to ensure a certain accuracy.

Let us now assume that $\bar{u}$ itself has the Slater property

$$S\bar{u} < \psi - \tau.$$

In contrast to Section 4 we do not require an inner point property with respect to the control constraints.

Theorem 5.2. *Let S fulfill assumption (9) and let $\bar{u}$ be a solution of (P) such that the Slater condition $S\bar{u} < \psi - \tau$ and a source condition $\bar{u} = P_{U_{\mathrm{ad}}}(S^*w)$ are fulfilled. Then it holds for sufficiently small λ and α that*

$$u_\alpha^0 = u_\alpha^\lambda.$$

Proof. From Theorem 3.1 we know the convergence of $\bar{u}_\alpha^0$ to $\bar{u}$ in U. Using the mapping property $S : U = L^2(D) \to L^\infty(D')$, we can show as in the proof of of Theorem 4.1 that

$$Su_\alpha^0 < \psi - \tau/2.$$

holds for sufficiently small α. From Lemma 5.1 we know that u_α^λ tends to u_α^0 for $\lambda \to 0$. Due to the properties of S we find

$$Su_\alpha^\lambda < \psi$$

for sufficiently small λ. For $\lambda \le \frac{\tau}{2b}$ we get

$$Su_\alpha^0 + \lambda u_\alpha^0 < \psi.$$

Consequently, u_α^0 is feasible for the problem (P_α^λ) and u_α^λ is feasible for the problem (P_α^0) for α and λ small enough. Testing the optimality condition for (P_α^λ) and (P_α^0) with u_α^0 and u_α^λ, respectively, yields, similar to the proof of Proposition 2.4, that $u_\alpha^\lambda = u_\alpha^0$. $\square$

Let us remark that the Lavrentiev regularization is also used with different sign, i.e.,

$$Su - \lambda u \le \psi.$$

Then, the set of admissible controls becomes larger for larger λ. Hence, the Slater condition ensures the existence of feasible controls for arbitrary positive λ. In this case we have no smallness condition for λ. The general result for the Lavrentiev regularization can be found in [3, Theorem 3.3]. The discussion for the specific case of Theorem 5.2 can be done completely analogue.

Both approaches have their specific advantages. As we have already seen, the first approach (plus sign) yields solutions that are feasible for the unregularized minimization problem. However, we have to deal with a smallness condition for

the Lavrentiev parameter λ. The second approach (minus sign) does not need an additional smallness condition, but the solutions are in general not feasible.

Finally we combine the results of Theorem 3.1, Theorem 4.1 and Theorem 5.2 to obtain an estimate for the total error of joint Tikhonov-Lavrentiev regularization under strong assumptions:

Theorem 5.3. *Let S fulfills the assumptions (5) and (6) and let $\bar{u}$ be a solution of (P) such that there exists $\tau > 0$ such that $\tau < \bar{u} < b - \tau$, $S\bar{u} < \psi - \tau$ and a $w \in Y$ such that $\bar{u} = S^* w$. Then it holds for λ small enough that the solution u_α^λ of (P_α^λ) fulfills*

$$\|\bar{u} - u_\alpha^\lambda\| = \mathcal{O}(\sqrt{\alpha}) \quad \textit{for} \quad \alpha \to 0.$$

Proof. We split the total error as

$$\|\bar{u} - u_\alpha^\lambda\| \leq \|\bar{u} - u_\alpha^0\| + \|u_\alpha^0 - u_\alpha^\lambda\|$$

and observe that due to Theorem 5.2 the second term vanishes if λ is small enough and that the first term behaves like $\mathcal{O}(\sqrt{\alpha})$ due to Theorem 3.1 and Theorem 4.1. □

6 Verification of the assumption in special cases

6.1 An elliptic control problem

In this section we will discuss the example from the introduction: Let $S : u \mapsto y$ be the solution operator of the boundary value problem

$$-\Delta y = u \text{ on } D, \quad y = 0 \text{ on } \partial D$$

with a Lipschitz domain $D \subset \mathbf{R}^N$, $N \in \{2, 3\}$. We specify the spaces $U = Y = L^2(D)$. The set $D' \subset D$ is assumed to be an inner subdomain. Let us now check the assumptions:

1. By the Lax-Milgram Lemma we obtain easily the existence of a unique solution $y \in H_0^1(D)$ for every $u \in U$. Due to the embedding $H_0^1(D) \hookrightarrow Y$, the operator S is well defined

2. Let us first mention that the operator S is selfadjoint for our specific choice of spaces. For the uniform boundedness of solutions of the elliptic problem we refer to Stampacchia [12].

3. The mapping property in Theorem 4.1 that $y_n \rightharpoonup y$ in Y implies $S^* y_n \to S^* y$ in $L^\infty(D)$ can be obtained by the following argumentation. Weak convergence $y_n \rightharpoonup y$ in $Y = L^2(D)$ implies strong convergence in $y_n \to y$ in $W^{-1,p}(D)$ for $2 \leq p < \infty$ for $N = 2$ and $2 \leq p \leq 6$ for $N = 3$. Now the desired result follows again from [12, Theorem 4.2].

4. The operator S has dense range in Y. That can be verified by the following argumentation. The space $C_0^\infty(D)$ is dense in Y. Moreover, all these functions belong to the image of S.

5. The Slater condition can only be checked (analytically or numerically) if one specifies the data. The Slater property for $\bar{u}$ is like the source condition an a priori assumption for the solution.

Consequently, we can apply all results of our paper to that example. Only the Slater property and the source condition are a priori assumptions. All other assumptions of that paper are satisfied for our example.

6.2 A Fredholm integral operator

Another class of examples in which the properties 1.–4. are fulfilled and which models several inverse problems is given by Fredholm integral operators [7] or [4, Chapter VI.]. Let D be a bounded domain in $\mathbf{R}^N$ and let $U = Y = L^2(D)$. For a Lipschitz continuous function $k : D \times D \to \mathbf{R}$ we consider $Su(x) = \int_D k(x,x')u(x')dx'$. Let us check the assumptions:

1. Since D is bounded, continuity of k implies that $S : L^2(D) \to L^2(D)$ compactly.

2. The uniform convergence of Su_n to Su follows from $u_n \to u$ by

$$\begin{aligned} \|Su_n - Su\|_\infty &\le \sup_{x\in D} \int_D |k(x,x')||u_n(x') - u(x')|dx' \\ &\le \sup_{x,y\in D} |k(x,y)|\sqrt{|D|}\|u_n - u\|_2. \end{aligned}$$

3. Uniform convergence of S^*y_n to S^*y follows from $y_n \rightharpoonup y$ by observing that $\|y_n - y\|_2$ is bounded and that the mapping $x \mapsto \int_D k(x',x)y(x')dx'$ is Lipschitz continuous for every $y \in L^2(D)$.

4. Since S is compact is has a representation via its singular value decomposition $Su = \sum_n \sigma_n \langle u, \psi_n\rangle \phi_n$ with non-negative singular values which decay to zero. We see that the range of S is dense if S does not have zero as a singular value (i.e. it is injective).

Again, the Slater condition and the source condition are a priori assumption for the solution.

Conclusion

In this paper we studied the simultaneous Tikhonov and Lavrentiev regularization of an optimal control problem with control and state constraints. We derived error estimates in the general case and the main tool was a source condition which resembles the classical one used in the inverse-problem context. With the help of this error estimate we could prove, under additional assumptions, that for sufficiently small Tikhonov parameter the Tikhonov-regularized solution does not have active inequality constraint if the original solution has the same property. Moreover, it was shown that for the Tikhonov- and Lavrentiev-regularized problem the solution coincides with the Tikhonov-regularized solution for sufficiently small regularization parameters if the unregularized solution has a Slater property. One may conclude that using additional physically motivated inequality constraints in the context of the regularization of inverse problems is a good idea for two reasons: For larger regularization parameters they may yield reconstructions that are more meaningful and asymptotically they do not destroy optimal convergence rates (and even become inactive if they will

be in the limit). On the other hand, additional inequality constraints lead to minimization problem which may be harder to solve. However, there are powerful algorithms available which allow a numerical treatment of such problems (cf. [8, 6]).

Moreover, our results provided a little insight in the problem of error estimates for joint Tikhonov-Lavrentiev regularization for optimal control problems and obtained preliminary estimates. Further research on the interpretation of the source condition and the Slater condition in particular contexts seems necessary. Finally we showed that the main assumptions for our setting are fulfilled for a distributed elliptic control problem and for some Fredholm equations of the first kind.

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